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Fourier Effectiveness and Order Summability*

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INTRODUCTION

We consider the Fourier expansion

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx} = \sum_{n=0}^{\infty} a_n(x)$$

at a point of continuity, say $x = 0$. It is well known that the partial sums of the series $\sum_{n=0}^{\infty} a_n(0)$ are $o(\log n)$, and that the series is summable by certain matrix methods (e.g., C_1, by Fejèr's theorem) which are called Fourier-effective (see definition in Section 1). In this paper we ask how much information is contained in the totality of all of these summability properties; in other words, we shall try to describe the intersection of the corresponding summability fields.

The series $\sum_{n=0}^{\infty} a_n(0)$ under consideration form a class F_C, and the summability field of an effective method B will be denoted by (B). We prove (Theorem 2.1) that

$$\bigcap (B) = F_C ,$$

i.e., that F_C can be characterized in terms of summability if we allow all effective methods. These include also nonregular methods, and we shall characterize them completely by properties of their kernels (Theorem 1.1).

There is a class of effective methods which are much better understood, and which we shall call monotone methods (Section 3). In the triangular case, Nikol'skiĭ [5] has completely characterized these methods by properties of

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the matrix elements (see Theorem 1.2). Again we ask for an explicit description of

$$\bigcap (A) = (L_1{}^*),$$

where we allow for A all effective monotone methods. It is the main result of this paper that $(L_1{}^*)$ can be defined directly, and in comparatively simple terms, through a certain summability method which will also be denoted by $L_1{}^*$. Since most methods ever used in this connection are either monotone or closely related to monotone ones practically all the information on summability known so far is contained in the theory of $L_1{}^*$. For this reason, it will be of interest to prove $L_1{}^*$-summability directly (Theorem 4.1). From the explicit definition it appears that $L_1{}^*$-summability of $\{s_n\}$ lies somewhere between the order-restriction $s_n = o(\log n)$ and C_1-summability; hence the name "order summability" for it. Actually, $L_1{}^*$ is equivalent to some monotone method (see Theorem 5.1) which simplifies the logical interrelation to a great extent. This and further implications will be discussed in Sections 5 and 6.

From the point of view of summability, effectiveness is closely related to inclusion theorems. In order to distinguish clearly between theorems concerning Fourier series and theorems which are of general interest in the summability theory, we have postponed the detailed study of these inclusion theorems to a paper following this ("Inclusion theorems and Order Summability"). Some results of this second paper will be used, however, in the discussion of implications in Section 6 of the present article.

1. Fourier Effectiveness

Let $f \in L[-\pi, \pi]$ have the Fourier expansion

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx} \equiv c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx}) \equiv \sum_{n=0}^{\infty} a_n(x).$$

The summability behavior of this series at a given point x is related to properties of $\varphi_x(t) = \frac{1}{2}(f(x+t) + f(x-t)) \in L[0, \pi]$ at $t = 0$, by virtue of the cosine-expansion

$$\varphi_x(t) \sim \sum_{n=0}^{\infty} a_n(x) \cos nt.$$

By F_C we denote the class of all series $\sum a_n(x)$ for which $\varphi_x(t)$ is continuous at $t = 0$, i.e., $\frac{1}{2}(f(x+t) + f(x-t)) \to \varphi_x(0) = f(x)$ as $t \to 0$. Similarly,

F_L is the class of all series $\sum a_n(x)$ for which $t = 0$ is an L-point of φ_x, i.e., $\int_0^h |\varphi_x(t) - \varphi_x(0)|\,dt = o(h)$ as $h \to +0$. Here x is to be considered as arbitrary but fixed, and will often be omitted. Thus, the class F_C (F_L) consists of all the series $\sum a_n$ for which $\sum a_n \cos nt$ is the cosine-expansion of a function $\varphi \in L[0, \pi]$ having $t = 0$ as point of continuity (as an L-point).

We consider summability methods $B = (b_{n\nu})$ in the series-to-sequence form satisfying

$$b_{n\nu} \to 1 \qquad (n \to \infty,\ \nu \text{ fixed}), \tag{1.1}$$

$$b_{n\nu} \to 0 \qquad (n \text{ fixed},\ \nu \to \infty). \tag{1.2}$$

(A method satisfying (1.1) and (1.2) need not be regular.) Further, we require that $\sigma_n(\varphi) = \sum_{\nu=0}^{\infty} b_{n\nu} a_\nu$ ($n = 0, 1,...$) is well defined (in some sense) for all series in F_C, resp. in F_L. It turns out that it is convenient to require here C_1-summability of $\sum b_{n\nu} a_\nu$, i.e.,

$$\sigma_n(\varphi) = \sum_{\nu=0}^{\infty} b_{n\nu} a_\nu \ (C_1), \qquad n = 0, 1,..., \tag{1.3}$$

for all series in F_C, resp. in F_L. This is the *applicability* condition. Finally, we require that

$$\sigma_n(\varphi) \to s = \varphi(0) \qquad (n \to \infty) \tag{1.4}$$

for all φ corresponding to series in F_C, resp. F_L. This is the *summability* condition (we note that (1.1) is a consequence of (1.4)). We write $\sum a_\nu = s$ (B) to indicate applicability and summability to s.

A method B satisfying (1.1), (1.2), (1.3) and (1.4) is called *Fourier-effective*, more precisely, *F_C-effective*, resp. *F_L-effective*. (It would be possible to define other types of effectiveness based upon a different local behavior of φ.)

The following theorem gives a characterization of F_C-effective methods.

THEOREM 1.1. *A method $B = (b_{n\nu})$ is F_C-effective if and only if*

$$\tfrac{1}{2} b_{n0} + \sum_{\nu=1}^{\infty} b_{n\nu} \cos \nu t \qquad (n = 0, 1,...)$$

are the cosine-expansions of functions (kernels) $b_n \in L[0, \tau]$ satisfying for every δ, $0 < \delta < \pi$:

$$\operatorname*{ess.\,sup}_{t \in [\delta, \pi]} |b_n(t)| \leqslant M_\delta \qquad (n = 0, 1,...), \tag{1.5}$$

$$\int_0^\pi |b_n(t)|\,dt \leqslant M \qquad (n = 0, 1,...), \tag{1.6}$$

$$\int_\delta^\pi b_n(t)\,dt \to 0, \qquad \frac{2}{\pi}\int_0^\pi b_n(t)\,dt \to 1 \quad (n \to \infty). \tag{1.7}$$

A corresponding result for F_L-effectiveness is not known in this generality. Since $F_C \subseteq F_L$, trivially F_L-effectiveness implies F_C-effectiveness. In particular, (1.6) is a necessary condition for Fourier-effectiveness. If B is regular, then (1.5) and (1.7) are automatically satisfied, and this case of Theorem 1.1 is known (see [1, Chapter VII, Section 2; 7, III, Section 2; 2]). Later on, however, we shall need Theorem 1.1 in its full generality.

Proof. Suppose that B is F_C-effective, and let first $\varphi \in C[0, \pi]$. Since a_ν depends linearly and continuously upon φ, so does $\sigma_n(\varphi)$ by Banach's limit theorem. Hence, there exists $B_n \in V[0, \pi]$ such that

$$\sigma_n(\varphi) = \frac{2}{\pi} \int_0^\pi \varphi(t)\, dB_n(t) \quad \text{for} \quad \varphi \in C[0, \pi], \qquad n = 0, 1, \ldots .$$

Here $B_n(t)$ may be normalized by $B_n(t) = B_n(t - 0)$ for $t \in (0, \pi)$. If we take $\varphi \in L_1[\delta, \pi]$ with the trivial extension $\varphi = 0$ on $[0, \delta)$, $\sigma_n(\varphi)$ is again linear and continuous and hence of the form

$$\sigma_n(\varphi) = \frac{2}{\pi} \int_\delta^\pi \varphi(t)\, b_{n,\delta}(t)\, dt, \qquad b_{n,\delta} \in L_\infty[\delta, \pi].$$

If $\varphi \in C[\delta, \pi]$, $\varphi(\delta) = 0$, we have both

$$\sigma_n(\varphi) = \frac{2}{\pi} \int_\delta^\pi \varphi(t)\, dB_n(t) = \frac{2}{\pi} \int_\delta^\pi \varphi(t)\, b_{n,\delta}(t)\, dt;$$

hence

$$B_n(t) = c_{n,\delta} + \int_\delta^t b_{n,\delta}(x)\, dx, \qquad t \in (\delta, \pi].$$

Since $\delta \in (0, \pi)$ is arbitrary, we conclude

B_n is absolutely continuous on $(0, \pi]$,

$B_n' = b_n \in L_\infty[\delta, \pi]$ and $\in L_1[0, \pi]$,

$$\sigma_n(\varphi) = c_n \varphi(0) + \frac{2}{\pi} \int_0^\pi \varphi(t)\, b_n(t)\, dt, \qquad \varphi \in C[0, \pi], \tag{1.8}$$

$$\sigma_n(\varphi) = \frac{2}{\pi} \int_\delta^\pi \varphi(t)\, b_n(t)\, dt, \qquad \varphi \in L_1[\delta, \pi]. \tag{1.9}$$

From the convergence $\sigma_n(\varphi) \to \varphi(0)$ it follows by the Banach-Steinhaus theorem that, for $n \to \infty$,

$$\int_0^\pi | b_n(t)|\, dt = O(1),$$

$$\operatorname*{ess.\,sup}_{t \in [\delta, \pi]} | b_n(t)| = O(1).$$

Also, from (1.9) it follows directly that

$$\int_\delta^\pi b_n(t)\,dt = o(1).$$

For $\varphi(t) = \cos \nu t$ it follows from (1.8) that

$$b_{n\nu} = c_n + \frac{2}{\pi}\int_0^\pi b_n(t)\cos \nu t\,dt.$$

Letting $\nu \to \infty$, we find $c_n = 0$ because of (1.2). Thus b_n is the kernel as explained in Theorem 1.1, and (1.8) yields

$$\frac{2}{\pi}\int_0^\pi b_n(t)\,dt \to 1.$$

This shows that the conditions of Theorem 1.1 are necessary.

Conversely, suppose that the kernel b_n has the properties (1.5), (1.6) and (1.7). Let $\varphi \in L_1[0, \pi]$ be continuous at $t = 0$ so that, in particular, $\varphi \in L_\infty[0, \delta_0]$. Consider the C_1-means of its cosine-expansion:

$$\varphi_k(t) = \sum_{\nu=0}^{k}\left(1 - \frac{\nu}{k+1}\right) a_\nu \cos \nu t$$

and note that for fixed $\delta_1 \in (0, \delta_0)$ and $k \to \infty$,

$$\sup_{t\in[0,\delta_1]} |\varphi_k(t)| = O(1),$$

$$\int_0^\pi |\varphi_k(t) - \varphi(t)|\,dt \to 0,$$

$$\varphi_k(t) \to \varphi(t) \text{ almost everywhere.}$$

By Lebesgue's theorem on dominated convergence on $[0, \delta_1]$ and by trivial estimates on $[\delta_1, \pi]$ it follows that, for $k \to \infty$,

$$\sum_{\nu=0}^{k}\left(1 - \frac{\nu}{k+1}\right) b_{n\nu}a_\nu = \frac{2}{\pi}\int_0^\pi \varphi_k(t)\,b_n(t)\,dt \to \frac{2}{\pi}\int_0^\pi \varphi(t)\,b_n(t)\,dt,$$

in view of (1.5) and (1.6). Thus, we have a "mixed" case of Parseval's formula

$$\sigma_n(\varphi) = \frac{2}{\pi}\int_0^\pi \varphi(t)\,b_n(t)\,dt = \sum_{\nu=0}^{\infty} b_{n\nu}a_\nu \,(C_1),$$

which is the applicability condition. The summability condition now follows by standard arguments. Condition (1.1) is the special case of (1.4) with

$\varphi(t) = \cos \nu t$, while (1.2) follows from the fact that $b_{n\nu}$ are Fourier coefficients. Hence B is F_C-effective.

Besides the characterization of F_C-effective methods by properties of the kernel it seems desirable also to characterize effectiveness in terms of the matrix elements $b_{n\nu}$. In this direction the following result is useful.

THEOREM 1.2. *If B is F_C-effective (or F_L-effective), then*

$$\left| \sum_{\substack{\nu=0 \\ \nu \neq k}}^{2k} \frac{b_{n\nu}}{k-\nu} \right| \leqslant M \qquad (n = 0, 1,...; k = 1, 2,...) \tag{1.10}$$

with a constant M independent of n and k.

Proof. Let

$$p_k(t) = \sum_{m=1}^{k} \frac{1}{m} \{\cos(k-m)t - \cos(k+m)t\} = 2 \sin kt \sum_{m=1}^{k} \frac{\sin mt}{m}.$$

Since $p_k(t)$ is uniformly bounded in k and t, we obtain from (1.6),

$$\sum_{\substack{\nu=0 \\ \nu \neq k}}^{2k} \frac{b_{n\nu}}{k-\nu} = \frac{2}{\pi} \int_0^{\pi} p_k(t)\, b_n(t)\, dt = O(1),$$

which proves the theorem.

If B is triangular, then it follows from (1.10), for $k = n + 1$, that

$$\left| \sum_{\nu=0}^{n} \frac{b_{n\nu}}{n+1-\nu} \right| \leqslant M, \qquad n = 0, 1,... \tag{1.11}$$

(see [5]).

2. The Intersection of Summability Fields of all F_C-Effective Methods

THEOREM 2.1. *Let $\sum a_\nu$ be summable to (the same) s by all F_C-effective methods B. Then*

$$\sum_{\nu=0}^{\infty} a_\nu \cos \nu t$$

is the cosine-expansion of a function $\varphi \in L_1[0, \pi]$ which is continuous at $t = 0$, $\varphi(0) = s$. In other words, the series belongs to F_C.

This theorem shows that the class F_C is completely characterized by summability properties. It follows that F_C-effectiveness is not the same as F_L-effectiveness, in general.

Proof. Fix the series $\sum a_n$ and consider only kernels $b_n \in C[0, \pi]$. Take an arbitrary $b \in C[0, \pi]$ with a cosine-expansion

$$b(t) \sim \tfrac{1}{2}b_0 + \sum_{\nu=1}^{\infty} b_\nu \cos \nu t.$$

Suppose that B (with kernel b_n) is F_C-effective. If we replace b_0 by b and leave b_n $(n \geqslant 1)$ unaltered, then the new method is F_C-effective (conditions (1.5), (1.6) and (1.7) remain true). Hence, by applicability and Banach's limit theorem,

$$\sigma(b) = \sum_{\nu=0}^{\infty} b_\nu a_\nu \quad (C_1) \tag{2.1}$$

is a continuous linear functional of $b \in C[0, \pi]$ and, therefore, of the form

$$\sigma(b) = \frac{2}{\pi}\int_0^\pi b(t)\, d\Phi(t), \quad \Phi \in V[0, \pi], \quad \Phi(t) = \Phi(t-0) \qquad \text{for} \quad t \in (0, \pi). \tag{2.2}$$

Returning to B, it follows that

$$\sigma_n = \frac{2}{\pi}\int_0^\pi b_n(t)\, d\Phi(t) \qquad (n = 0, 1, \ldots).$$

Putting $\Phi_0(t) = \Phi(t) - st$, we know from the summability that

$$\int_0^\pi b_n(t)\, d\Phi_0(t) \to 0 \qquad (n \to \infty), \tag{2.3}$$

whenever $b_n \in C[0, \pi]$ satisfies (1.5), (1.6) and (1.7).

Consider an arbitrary $c_n \in C[0, \pi]$ satisfying

$$\sup_{t\in[0,\pi]} |c_n(t)| = O(1), \qquad \int_0^\pi |c_n(t)|\, dt \to 0 \quad (n \to \infty).$$

Since the kernel $b_n + c_n$ satisfies (1.5), (1.6) and (1.7), it follows that

$$\int_0^\pi c_n(t)\, d\Phi_0(t) \to 0 \qquad (n \to \infty).$$

If one uses c_n to approximate the characteristic function of a single point, one sees that $\Phi_0(t)$ is continuous on $[0, \pi]$. If one uses c_n to approximate the characteristic function of a nonoverlapping union of finitely many intervals of small total length, one sees that Φ_0 is absolutely continuous on $[0, \pi]$. In particular, with $\Phi'(t) = \varphi(t) \in L[0, \pi]$, (2.1) and (2.2) give

$$\frac{2}{\pi}\int_0^\pi \cos \nu t\, \varphi(t)\, dt = \begin{cases} 2a_0 & \text{for } \nu = 0, \\ a_\nu & \text{for } \nu \geqslant 1. \end{cases}$$

Furthermore, (2.3) takes the form

$$\int_0^\pi b_n(t)\, \varphi_0(t)\, dt = o(1) \qquad (\varphi_0(t) = \varphi(t) - s).$$

Next, we show that

$$\operatorname*{ess\,sup}_{t\in[0,\delta]} |\varphi_0(t)| \to 0 \qquad (\delta \to +0). \tag{2.4}$$

We restrict ourselves to kernels $b_n(t) \geqslant 0$ and may, therefore, assume that $\varphi_0(t)$ is real valued. If (2.4) fails, we may assume that $\epsilon > 0$ and $\delta_n \to +0$ exist such that $\varphi_0(t) > \epsilon$ holds on a subset $E_n \subset [0, \delta_n]$ with positive measure m_n. Let χ_n be the characteristic function of E_n and $d_n(t) = \pi\chi_n(t)/(2m_n)$. Then d_n satisfies (1.5), (1.6) and (1.7), but $\int_0^\pi d_n(t)\, \varphi_0(t)\, dt > \epsilon$. Now we approximate d_n by continuous functions $b_n \geqslant 0$ and obtain a contradiction. Hence (2.4) holds, and by changing φ_0 possibly on a set of measure zero we ensure $\varphi_0(t) \to 0 = \varphi_0(0)$ $(t \to +0)$. This concludes the proof of Theorem 2.1.

3. Positive (Monotone) Methods

Let $A = (a_{n\nu})$ be a matrix with the properties

$$a_{n\nu} \geqslant 0, \qquad a_{n\nu} \to 0 \quad (n \to \infty,\ \nu \text{ fixed}), \qquad \sum_{\nu=0}^{\infty} a_{n\nu} \to 1 \quad (n \to \infty). \tag{3.1}$$

It follows from (3.1) that A defines a regular sequence-to-sequence transformation. We associate with it a series-to-sequence method $B = (b_{n\nu})$ defined by

$$b_{n\nu} = \sum_{\mu=\nu}^{\infty} a_{n\mu} \qquad (n, \nu = 0, 1, \ldots). \tag{3.2}$$

The matrix B satisfies (1.1) and (1.2), and also

$$b_{n\nu} \downarrow 0 \qquad (n \text{ fixed},\ \nu \to \infty). \tag{3.3}$$

From $\sum_{\nu=0}^{k} a_{n\nu}s_\nu = -s_k b_{n,k+1} + \sum_{\nu=0}^{k} b_{n\nu}a_\nu$, $s_n = \sum_{\nu=0}^{n} a_\nu$, we obtain, for every C_1-summable series $\sum a_\nu$ (e.g., for every series of F_C or F_L), the relation

$$\sum_{\nu=0}^{\infty} a_{n\nu}s_\nu = \sum_{\nu=0}^{\infty} b_{n\nu}a_\nu \quad (C_1) \tag{3.4}$$

since $s_k b_{n,k+1} \to 0(C_1)$ because of (3.3). In (3.4), the C_1-summability of one series implies that of the other. Thus, for those series, $s_n \to s\,(A)$ is equivalent to $\sum a_\nu = s\,(B)$ including the applicability (C_1) in both cases. A method A is called F_C-effective (resp. F_L-effective) if the summability field (A) contains F_C (resp. F_L) with $s = \varphi(0)$, which is equivalent to B being F_C-effective (resp. F_L-effective). By Theorem 1.1 we see that A is F_C-effective if and only if

$$\int_0^\pi |\, b_n(t)|\, dt = O(1), \tag{3.5}$$

where, in view of (3.2) and (3.3),

$$b_n(t) = \tfrac{1}{2}b_{n0} + \sum_{\nu=1}^{\infty} b_{n\nu}\cos \nu t = \sum_{\nu=0}^{\infty} a_{n\nu}\frac{\sin(\nu+\frac{1}{2})t}{2\sin\frac{t}{2}}$$

for $t \in (0, \pi]$, $n = 0, 1,\ldots$. The necessary condition (1.10) can be written in the form

$$\left|\sum_{m=1}^{k}\frac{1}{m}\sum_{\nu=k-m}^{k+m-1} a_{n\nu}\right| \leqslant M \qquad (n \geqslant 0, k \geqslant 1). \tag{3.6}$$

A method A will be called *monotone*, if in addition to (3.1), a sequence of integers $\nu_n \geqslant 0$ exists such that

$$a_{n\nu}\uparrow \quad (0 \leqslant \nu \leqslant \nu_n), \qquad a_{n\nu}\downarrow \quad (\nu_n \leqslant \nu), \qquad \text{for} \quad \nu\uparrow. \tag{3.7}$$

If A is monotone, then for $n \to \infty$,

$$\nu_n a_{n0} = O(1), \qquad \sum_{\nu=0}^{\infty} |\,\nu - \nu_n\,|\,|\,\Delta a_{n\nu}\,| = O(1) \quad (\Delta a_{n\nu} = a_{n\nu} - a_{n,\nu+1}). \tag{3.8}$$

This follows immediately from

$$\sum_{\nu=0}^{K} |\,\nu-\nu_n\,|\,|\,\Delta a_{n\nu}\,| = \sum_{\nu=0}^{K} (\nu-\nu_n)\,\Delta a_{n\nu}$$

$$= -\nu_n a_{n0} + \sum_{\nu=1}^{k} a_{n\nu} - (K-\nu_n)\,a_{n,K+1} \leqslant \sum_{\nu=1}^{K} a_{n\nu}$$

$$(K \geqslant \nu_n).$$

4. Order Summability

In this section we shall prove an estimate for the partial sums of the series belonging to F_L, and this result will lead to the definition of summability $L_1{}^*$.

THEOREM 4.1. *Let $\sum a_n \in F_L$, and let $s_n = \sum_{\nu=0}^{n} a_\nu$. Then*

$$\frac{1}{n+1-m} \sum_{\nu=m}^{n} (s_\nu - \varphi(0)) = o\left(1 + \log \frac{n+1}{n+1-m}\right) \tag{4.1}$$

as $n \to \infty$, uniformly for $0 \leqslant m \leqslant n$.

Proof. A short calculation shows that

$$\begin{aligned} S_{mn} &= \sum_{\nu=m}^{n} (s_\nu - \varphi(0)) \\ &= \frac{1}{\pi} \int_0^\pi (\varphi(t) - \varphi(0)) \frac{\sin(n+1-m)\frac{t}{2} \sin(n+1+m)\frac{t}{2}}{\sin^2 \frac{t}{2}}\, dt. \end{aligned} \tag{4.2}$$

It follows from (4.2) and $|\sin y| \leqslant \min(|y|, 1)$, $1/\sin(t/2) \leqslant \pi/t$ $(0 < t \leqslant \pi)$, that

$$\begin{aligned} &\frac{|S_{mn}|}{\pi(n+1-m)} \\ &\quad \leqslant \frac{n+1+m}{4} \int_0^{1/(n+1+m)} |\varphi(t) - \varphi(0)|\, dt + \int_{1/(n+1+m)}^{1/(n+1-m)} |\varphi(t) - \varphi(0)| \frac{dt}{2t} \\ &\quad + \frac{1}{n+1-m} \int_{1/(n+1-m)}^{\pi} |\varphi(t) - \varphi(0)| \frac{dt}{t^2}. \end{aligned} \tag{4.3}$$

Introducing by partial integration the function $\rho(t) = (1/t) \int_0^t |\varphi(t) - \varphi(0)|\, dt$ the right side of the inequality (4.3) is at most

$$\frac{\rho(\pi)}{\pi(n+1-m)} + \int_{1/(n+1+m)}^{1/(n+1-m)} \frac{\rho(t)}{t}\, dt + \frac{2}{n+1-m} \int_{1/(n+1-m)}^{\pi} \frac{\rho(t)}{t^2}\, dt. \tag{4.4}$$

If $n \to \infty$, $n - m \to \infty$, then (4.4) is (uniformly)

$$o(1) + o\left(\log \frac{n+1+m}{n+1-m}\right) + o(1) = o\left(1 + \log \frac{n+1}{n+1-m}\right).$$

If $n \to \infty$, $n - m = O(1)$, then (4.4) is (uniformly)

$$O(1) + o(\log n) + O(1) = o\left(1 + \log \frac{n+1}{n+1-m}\right).$$

This proves (4.1).

A sequence $\{s_n\}$ is called L_1^*-*summable* to s, we write $s_n \to s$ (L_1^*), if

$$\frac{1}{n+1-m} \sum_{\nu=m}^{n} (s_\nu - s) = o\left(1 + \log \frac{n+1}{n+1-m}\right) \tag{4.5}$$

as $n \to \infty$, uniformly for $0 \leqslant m \leqslant n$. Using this definition and denoting the summability field of L_1^* by (L_1^*), Theorem 4.1 reads $F_L \subseteq (L_1^*)$. We note that $s_n \to s$ (L_1^*) implies $s_n = o(\log n)$ (take $m = n$), and $s_n \to s$ (C_1) (take $m = 0$).

We generalize the foregoing definition. Suppose that $g(t)$ is defined for $t \in [0, 1)$ and that $g(t) \geqslant 0$. Then a sequence $\{s_n\}$ is called *order-summable* $[g]$ to s, and we write $s_n \to s$ $[g]$ if

$$\frac{1}{n+1-m} \sum_{\nu=m}^{n} (s_\nu - s) = o\left(1 + g\left(\frac{m}{n+1}\right)\right) \tag{4.6}$$

as $n \to \infty$, uniformly for $0 \leqslant m \leqslant n$.

It follows from (4.5) and (4.6) that order-summability $[\log 1/(1 - t)]$ is summability L_1^*. Furthermore, it follows from (4.6) that $s_n \to s$ $[g]$ implies $s_n - s = o(1 + g(1 - 1/(n + 1)))$ and $s_n \to s$ (C_1).

5. Equivalence

In this section we shall show that order summability can also be expressed as ordinary summability by a triangular and monotone method. This observation is of theoretical interest (see Introduction). From a technical standpoint it is in most cases more convenient to use the notion of order summability rather than the corresponding matrix method.

THEOREM 5.1. *Given g, there is a monotone and triangular method A^* which is equivalent to $[g]$.*

Proof. Define for $0 \leqslant m \leqslant n$, $\nu \geqslant 0$:

$$a_{nm}^*(\nu) = \begin{cases} \dfrac{1}{n+1} \dfrac{g\left(\dfrac{m}{n+1}\right)}{1+g\left(\dfrac{m}{n+1}\right)} & \text{if } \nu < m, \\ \dfrac{1}{n+1} \dfrac{g\left(\dfrac{m}{n+1}\right)}{1+g\left(\dfrac{m}{n+1}\right)} + \dfrac{1}{n+1-m} \dfrac{1}{1+g\left(\dfrac{m}{n+1}\right)} & \\ & \text{if } m \leqslant \nu \leqslant n, \\ 0 & \text{if } \nu > n. \end{cases}$$

In particular,

$$a_{n0}^*(\nu) = \begin{cases} \dfrac{1}{n+1} & \text{if } 0 \leqslant \nu \leqslant n, \\ 0 & \text{if } \nu > n. \end{cases} \tag{5.1}$$

Clearly, $a_{nm}^*(\nu) \geqslant 0$, $\sum_{\nu=0}^{\infty} a_{nm}^*(\nu) = 1$, and for $\nu \uparrow$ and arbitrary integers $\nu_{nm} \in [m, n]$,

$$a_{nm}^*(\nu) \uparrow \quad (0 \leqslant \nu \leqslant \nu_{nm}), \qquad a_{nm}^*(\nu) \downarrow \quad (\nu \geqslant \nu_{nm}).$$

Furthermore, $a_{nm}^*(\nu) \to 0$ for fixed ν, $n \to \infty$ uniformly in $0 \leqslant m \leqslant n$.

Now arrange the pairs (n, m) in lexicographic order so that (n, m) is the k-th pair, where $k = n(n+1)/2 + m$. This defines $a_{k\nu}^* = a_{nm}^*(\nu)$, and the matrix A^* is monotone and triangular. Obviously, $\sum_{\nu=0}^{k} a_{k\nu}^* s_\nu \to 0$ is equivalent to

$$\sum_{\nu=0}^{n} a_{nm}^*(\nu)\, s_\nu = \frac{g\left(\dfrac{m}{n+1}\right)}{1+g\left(\dfrac{m}{n+1}\right)} \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu + \frac{1}{1+g\left(\dfrac{m}{n+1}\right)} \frac{1}{n+1-m} \sum_{\nu=m}^{n} s_\nu \to 0$$

as $n \to \infty$ uniformly in $0 \leqslant m \leqslant n$, and in view of (5.1), both are equivalent to $s_n \to 0$ $[g]$.

6. Implications

In conclusion, we discuss some implications of the foregoing results.

For monotone methods, (3.6) implies (3.5), as can be shown by direct estimates. Thus, *for monotone methods*, (3.6) *is equivalent to F_C-effectiveness.*

However, we shall show in the succeeding paper that (3.6), for monotone methods, is also equivalent to $A \supseteq L_1^*$, which implies that A is even F_L-effective. It follows that, *for monotone methods, F_C-effectiveness is the same as F_L-effectiveness*; so we can simply speak of effectiveness (without qualification). Also it follows that, *for monotone methods, effectiveness is equivalent to the inclusion $A \supseteq L_1^*$*. Here we may replace L_1^* by the equivalent method A^* (Theorem 5.1). Hence, *among the monotone effective methods A there is a weakest method A^*, in particular $\bigcap A = A^* \approx L_1^*$*. Since L_1^* is F_L-effective, the class F_C cannot be characterized by monotone effective methods only.

We shall also show in the above-mentioned following paper that the inclusion $A \supseteq L_1^*$ (with a regular, but not necessarily monotone A) stems from the following condition which is weaker than (3.6) for monotone methods: For a suitable sequence of integers $\nu_n \geqslant 0$,

$$\sum_{\nu \neq \nu_n} | \nu - \nu_n | \, g^* \left(\frac{\nu + 1}{\nu_n + 1} \right) | \Delta a_{n\nu} | = O(1),$$

where

$$g^*(t) = t \left(1 + \log \frac{1}{1 - t}\right) \qquad \text{for} \quad t \in [0, 1), \tag{6.1}$$

$$g^*(t) = 1 + \log \frac{t}{t - 1} \qquad \text{for} \quad t > 1.$$

Thus, this condition still implies F_C- and F_L-effectiveness, as was already observed for $\nu_n = n$ by Nagy [6], and, in general, by Karamata and Tomić [4].

Karamata [3] has shown that a monotone Nörlund mean N_p is effective if and only if $C_\epsilon \subseteq N_p$ for some $\epsilon > 0$. Thus, $\bigcap N_p = \bigcap_{\alpha > 0} C_\alpha$, when we allow all monotone and effective Nörlund methods. In the following paper we shall give another proof of this result. From the discussion of Wiener-type methods it will be apparent that monotone effective methods A exist satisfying $L_1^* \subsetneq A \subsetneq \bigcap_{\alpha > 0} C_\alpha$, with strict inclusion on both sides. Therefore, the set of all monotone effective Nörlund methods does in no way represent the class of all monotone effective methods. A similar result is true for monotone effective arithmetical means M_p : $\bigcap M_p = C_1$.

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